Mimetic Finite Difference Methods for Diffusion Equations on AMR grids

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Objectives

What do we want from the discretizations?

- preserve and mimic mathematical properties of physical systems;

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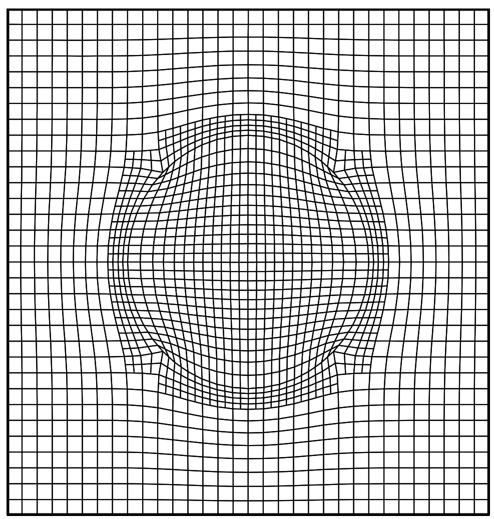
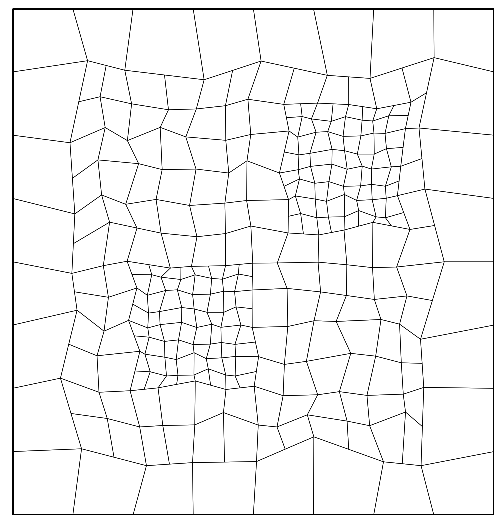
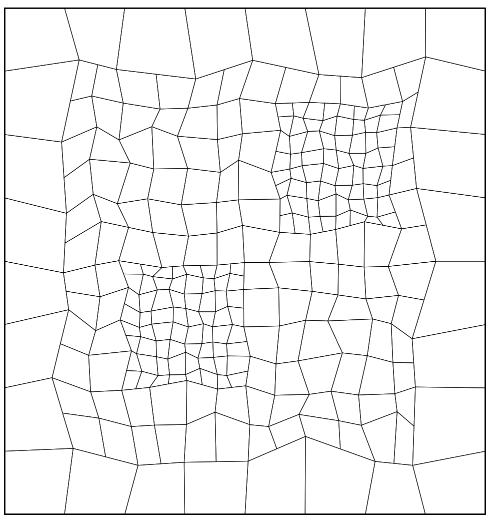


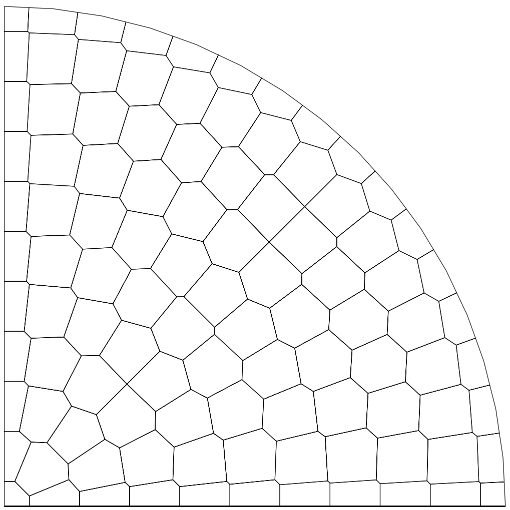


Objectives

What do we want from the discretizations?

- preserve and mimic mathematical properties of physical systems;
- be accurate on adaptive grids;
- be applicable to a large family of grids and operators.





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Mimetic discretizations (1/6)

Consider the mathematical identity:

$$\int_\Omega \mathrm{grad}\, p\, \boldsymbol{f}\, \mathrm{d}v = -\int_\Omega \mathrm{div}\, \boldsymbol{f}\, p\, \mathrm{d}v \qquad \forall \boldsymbol{f} \in H_{div}(\Omega),\ p \in H_0^1(\Omega).$$

Support-operators (SO) methodology (for div & grad):

1. define degrees of freedom for the physical variables (p, $\boldsymbol{f}$);
2. equip each of the discrete spaces with a scalar product ($[\cdot\,,\,\cdot]_Q$, $[\cdot\,,\,\cdot]_X$);
3. choose a discrete approximation to the divergence operator (the *prime* operator $\mathrm{DIV}: X_d \to Q_d$);
4. derive the discrete approximation of the gradient operator from the Green formula (the *derived* operator $\mathrm{GRAD}: Q_d \to X_d$) s.t. the following discrete identity is enforced:

$$[\boldsymbol{f}^d,\, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, \boldsymbol{f}^d,\, p^d]_Q \qquad \forall p^d \in Q_d,\ \boldsymbol{f}^d \in X_d.$$

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Mimetic discretizations (2/6)

Applications of the SO methodology include:

- Electromagnetics: discrete operators DIV, $\overline{\text{GRAD}}$, CURL and $\overline{\text{CURL}}$ mimic:

$$\text{div curl} = 0, \qquad \text{curl grad} = 0$$

$$\int_\Omega \text{curl}\boldsymbol{E}\cdot\boldsymbol{H}\,\mathrm{d}v = \int_\Omega \text{curl}\boldsymbol{H}\cdot\boldsymbol{E}\,\mathrm{d}v + \oint_{\partial\Omega}(\boldsymbol{E}\times\boldsymbol{H})\cdot\boldsymbol{n}\,\mathrm{d}s$$

- CFD: discrete operators DIV and GRAD mimic:

$$\int_\Omega \text{grad}\,\boldsymbol{u} : \boldsymbol{T}\,\mathrm{d}v = -\int_\Omega \text{div}\,\boldsymbol{T}\cdot\boldsymbol{u}\,\mathrm{d}v + \oint_{\partial\Omega}\boldsymbol{u}\cdot(\boldsymbol{T}\cdot\boldsymbol{n})\,\mathrm{d}s$$

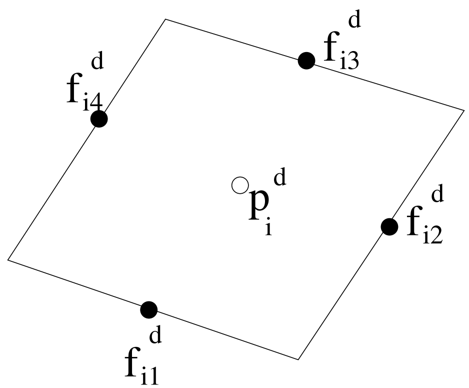
- Gas dynamics, poroelasticity, magnetic diffusion, etc...

Mimetic discretizations (3/6)

- Let Q_d be a vector space of cell-centered discrete scalar functions with the scalar product

$$[p^d,\, q^d]_Q = \sum_{i=1}^{N} |e_i|\, p_i^d\, q_i^d \qquad \forall p^d,\, q^d \in Q_d.$$

- Let X_d be a vector space of discrete edge-based vector functions with a scalar product $[\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X$. The vector function $\boldsymbol{f}^d$ is recovered exactly at four vertices of quadrilateral e_i. Let



$$[\boldsymbol{f}_i^d,\, \boldsymbol{g}_i^d]_{X,i} = \frac{1}{2}\sum_{j=1}^{4} |e_{ij}|\, K_i^{-1} \boldsymbol{f}_{ij}^d \cdot \boldsymbol{g}_{ij}^d$$

Then $\quad [\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X = \sum_{i=1}^{N} [\boldsymbol{f}_i^d,\, \boldsymbol{g}_i^d]_{X,i}.$

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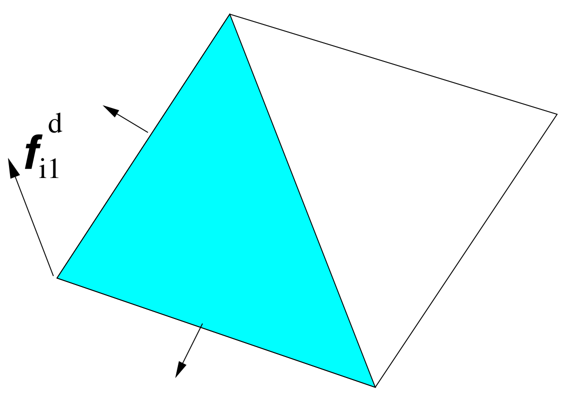

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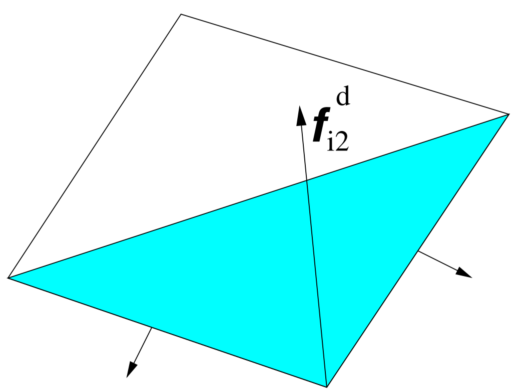

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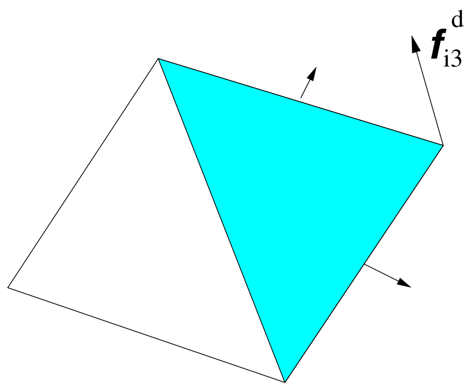

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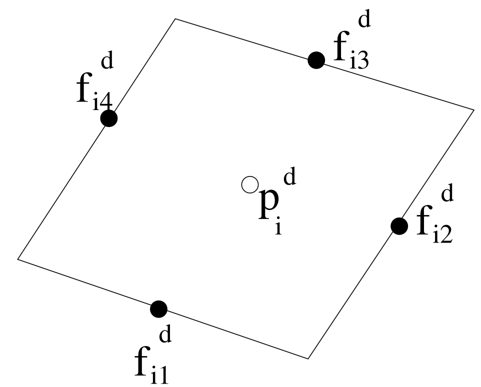
Mimetic discretizations (4/6)



The prime operator DIV is derived from the Gauss theorem:

$$\operatorname{div} \boldsymbol{f} = \lim_{|e|\to 0} \frac{1}{|e|} \oint_{\partial e} \boldsymbol{f} \cdot \boldsymbol{n}\, dl.$$

Center-point quadrature gives

$$\left(\mathrm{DIV}\, \boldsymbol{f}^d\right)_i = \frac{1}{|e_i|} \left(f_{i2}^d\, |l_2| - f_{i4}^d\, |l_4| + f_{i3}^d\, |l_3| - f_{i1}^d\, |l_1| \right)$$

The derived operator GRAD is implicitly given by

$$[\boldsymbol{f}^d, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, \boldsymbol{f}^d, p^d]_Q \qquad \forall p^d \in Q_d,\ \boldsymbol{f}^d \in X_d.$$

Mimetic discretizations (5/6)

The stationary diffusion problem

$$\begin{aligned} -\mathrm{div}\, K\nabla p &= b \quad \text{in} \quad \Omega \\ p &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

is rewritten as the 1st order system

$$\boldsymbol{f} = -K\nabla p, \qquad \mathrm{div}\, \boldsymbol{f} = b$$

and discretized as follows:

$$\boldsymbol{f}^d = -\mathrm{GRAD}\, p^d, \qquad \mathrm{DIV}\, \boldsymbol{f}^d = b^d.$$

Mimetic discretizations (6/6)

By the definition,

$$[\boldsymbol{f}^d,\, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, \boldsymbol{f}^d,\, p^d]_Q.$$

Let $< \cdot\,, \cdot >$ be the usual vector dot product. Then

$$[p^d,\, q^d]_Q =< \mathcal{D} p^d,\, q^d >, \qquad [\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X =< \mathcal{M} \boldsymbol{f}^d,\, \boldsymbol{g}^d > .$$

Combining the last two formulas, we get

$$\begin{aligned} [\boldsymbol{f}^d,\, \mathrm{GRAD} p^d]_X &= < \mathcal{M}\, \boldsymbol{f}^d,\, \mathrm{GRAD} p^d > \\ &= -[\mathrm{DIV}\, \boldsymbol{f}^d,\, p^d]_Q = - < \boldsymbol{f}^d,\, \mathrm{DIV}^t\, \mathcal{D}\, p^d > . \end{aligned}$$

Therefore,

$$\mathrm{GRAD} = -\mathcal{M}^{-1}\, \mathrm{DIV}^t\, \mathcal{D}.$$

is the non-local operator.

Connections with FE methods (1/5)

The system of finite difference equations

$$\boldsymbol{f}^d = -\mathrm{GRAD}\, p^d, \qquad \mathrm{DIV}\, \boldsymbol{f}^d = b^d$$

can be rewritten as

$$\begin{aligned} [\boldsymbol{f}^d, \boldsymbol{g}^d]_X + [\mathrm{GRAD} p^d, \boldsymbol{g}^d]_X &= 0, \\ [\mathrm{DIV}\, \boldsymbol{f}^d, q^d]_Q &= [b^d, q^d]_Q. \end{aligned}$$

Recall that by the definition,

$$[\boldsymbol{f}^d, \mathrm{GRAD}\, p^d]_X = -[\mathrm{DIV}\, \boldsymbol{f}^d, p^d]_Q.$$

Connections with FE methods (2/5)

Thus, the mimetic discretizations are equivalent to

$$\begin{aligned}
[\boldsymbol{f}^d, \boldsymbol{g}^d]_X - [\mathrm{DIV}\, \boldsymbol{f}^d, p^d]_Q &= 0, \\
-[\mathrm{DIV}\, \boldsymbol{f}^d, q^d]_Q &= -[b^d, q^d]_Q, \qquad \forall p^d \in Q_d,\ \boldsymbol{g}^d \in X_d.
\end{aligned}$$

On the other hand, the MFE method with the *Raviart-Thomas* elements gives

$$\begin{aligned}
(K^{-1}\boldsymbol{f}^h, \boldsymbol{g}^h) - (\mathrm{div}\, \boldsymbol{f}^h, p^h) &= 0, \\
-(\mathrm{div}\, \boldsymbol{f}^h, q^h) &= -(b, q^h) \qquad \forall q^h \in Q_h,\ \boldsymbol{g}^h \in X_h.
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Connections with FE methods (2/5)

Thus, the mimetic discretizations are equivalent to

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\begin{aligned}
[\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X - [\mathrm{DIV}\, \boldsymbol{f}^d,\, p^d]_Q &= 0, \\
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\end{aligned}
$$

Degrees of freedom:			
	p^d:	at cell centers	one per cell
	$\boldsymbol{f}^d$:	normal components at edge centers	normal compone

Connections with FE methods (3/5)

There are isomorphisms $\mathcal{I}_X$ and isometry $\mathcal{I}_Q$:

$$\mathcal{I}_X : X_d \to X_h, \qquad \mathcal{I}_Q : Q_d \to Q_h.$$

Properties:

- $[p^d,\, q^d]_Q = (p^h,\, q^h), \qquad p^h = \mathcal{I}_Q(p^d),\; q^h = \mathcal{I}_Q(q^d)$

- $[\mathrm{DIV}\, \boldsymbol{f}^d,\, p^d]_Q = (\mathrm{div}\, \boldsymbol{f}^h,\, p^h), \qquad \boldsymbol{f}^h = \mathcal{I}_X(\boldsymbol{f}^d)$

- $[\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X = (K^{-1}\boldsymbol{f}^h,\, \boldsymbol{g}^h) + O(h), \qquad \boldsymbol{g}^h = \mathcal{I}_X(\boldsymbol{g}^d)$

Therefore $[\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X$ may be considered as a quadrature rule for $(K^{-1}\boldsymbol{f}^h,\, \boldsymbol{g}^h)$.

Connections with FE methods (4/5)

Theorem.

Suppose $\mathcal{T}_h$ is either a shape regular *triangular* or a quasi-uniform *quadrilateral* partitioning of $\bar{\Omega}$ and input data are sufficiently smooth. Denote the solution of the finite difference method by $(\boldsymbol{f}^d, p^d)$, and set

$$\boldsymbol{f}^h = \mathcal{I}_X(\boldsymbol{f}^d), \qquad p^h = \mathcal{I}_Q(p^d).$$

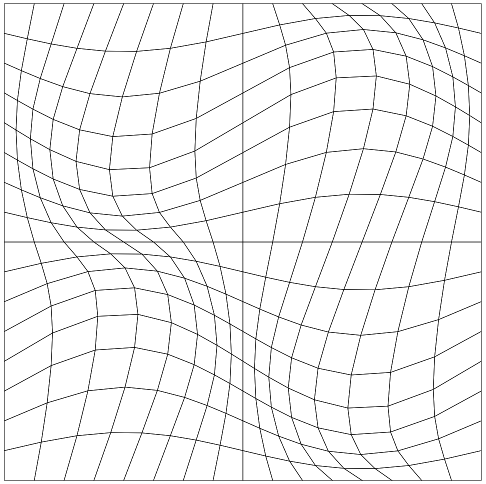
Then, the following error bounds hold

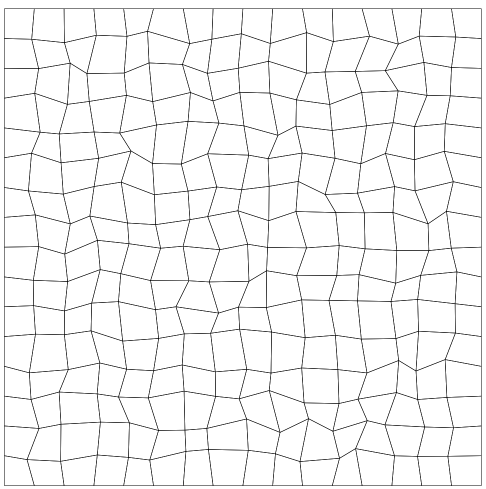
$$\|\boldsymbol{f} - \boldsymbol{f}^h\|_{\mathrm{div},\,\Omega} \le C\,h\,\left\{\,\|\boldsymbol{f}\|_1 + \|\mathrm{div}\,\boldsymbol{f}\|_1\,\right\},$$

$$\|p - p^h\|_{\Omega} \le C\,h\,\left\{\,\|p\|_1 + \|\boldsymbol{f}\|_1\,\right\},$$

with a positive constant C independent of h.

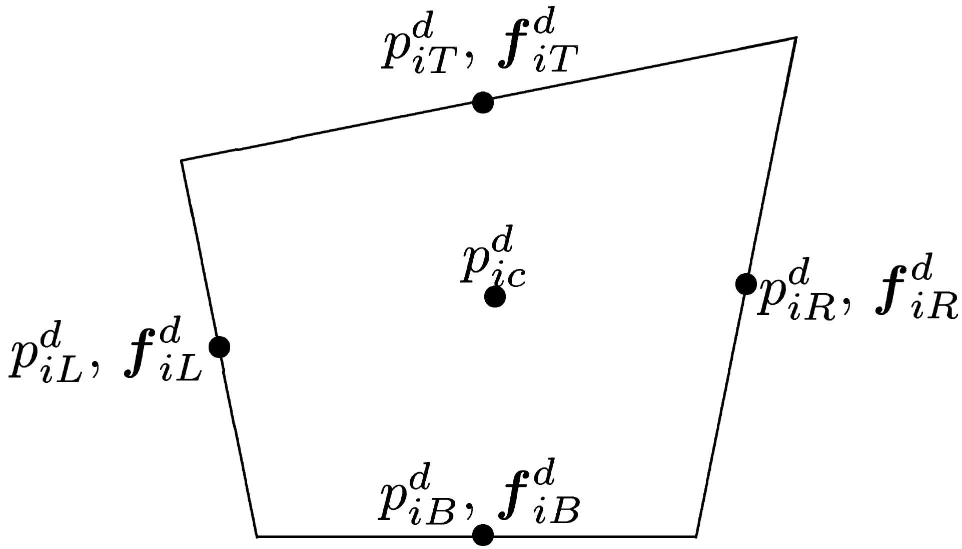
Connections with FE methods (5/5)





h^{-1}	modified RT FE		SO FD	
	ε_p	$\varepsilon_{\boldsymbol{f}}$	ε_p	$\varepsilon_{\boldsymbol{f}}$
16	1.58e-3	2.34e-2	1.61e-3	2.35e-2
32	7.95e-4	1.22e-2	7.99e-4	1.22e-2
64	3.98e-4	6.29e-3	3.99e-4	6.29e-3
128	1.99e-4	3.22e-3	1.99e-4	3.22e-3
256	9.97e-5	1.64e-3	9.97e-5	1.64e-3
512	4.98e-5	8.32e-4	4.98e-5	8.32e-4
	ε_p	$\varepsilon_{\boldsymbol{f}}$	ε_p	$\varepsilon_{\boldsymbol{f}}$
16	1.42e-3	2.24e-2	1.43e-3	2.25e-2
32	7.15e-4	1.17e-2	7.18e-4	1.17e-2
64	3.59e-4	5.96e-3	3.59e-4	5.98e-3
128	1.80e-4	3.06e-3	1.80e-4	3.07e-3
256	9.00e-5	1.56e-3	9.00e-5	1.56e-3
512	4.50e-5	7.93e-4	4.50e-5	7.93e-4

Mimetic discretizations (1/4)



The SO method mimic the mathematical identity

$$\int_e \boldsymbol{f} \cdot \mathrm{grad} p + \int_e \mathrm{div}\, \boldsymbol{f}\, p = \int_{\partial e} p\, \boldsymbol{f} \cdot \boldsymbol{n}.$$

Degrees of freedom:

p^d: at cell centers and edge centers

$\boldsymbol{f}^d$: normal components at edge centers

Mimetic discretizations (2/4)

The prime operator DIV is derived from the Gauss theorem:

$$(\mathrm{DIV}\,\boldsymbol{f}^d)_i = \frac{1}{|e_i|}\left(f^d_{iR}|l_{iR}| + f^d_{iT}|l_{iT}| + f^d_{iL}|l_{iL}| + f^d_{iB}|l_{iB}|\right)$$

Derivation of the discrete identity:

- $$\int_{e_i} \boldsymbol{f}\cdot \mathrm{grad} p\, \mathrm{d}x \approx [\boldsymbol{f}^d_i, (\mathrm{GRAD}\, p^d)_i]_{X_i}$$
- $$\int_{e_i} \mathrm{div}\, \boldsymbol{f}\, p\, \mathrm{d}x \approx (\mathrm{DIV}\,\boldsymbol{f}^d)_i\, p^d_i\, |e_i|$$
- $$\int_{\partial e_i} p\, \boldsymbol{f}\cdot \boldsymbol{n}\, \mathrm{d}s \approx p^d_{iR} f^d_{iR}|l_{iR}| + p^d_{iT} f^d_{iT}|l_{iT}| + p^d_{iL} f^d_{iL}|l_{iL}| + p^d_{iB} f^d_{iB}|l_{iB}|$$

Mimetic discretizations (3/4)

Replacing integrals in the Gauss-Green formula by their approximations, we get

$$(\mathrm{GRAD}\, p^d)_i = \mathcal{M}_i^{-1} \begin{pmatrix} |l_{iR}|(p^d_{iR} - p^d_{ic}) \\ |l_{iT}|(p^d_{iT} - p^d_{ic}) \\ |l_{iL}|(p^d_{iL} - p^d_{ic}) \\ |l_{iB}|(p^d_{iB} - p^d_{ic}) \end{pmatrix}$$

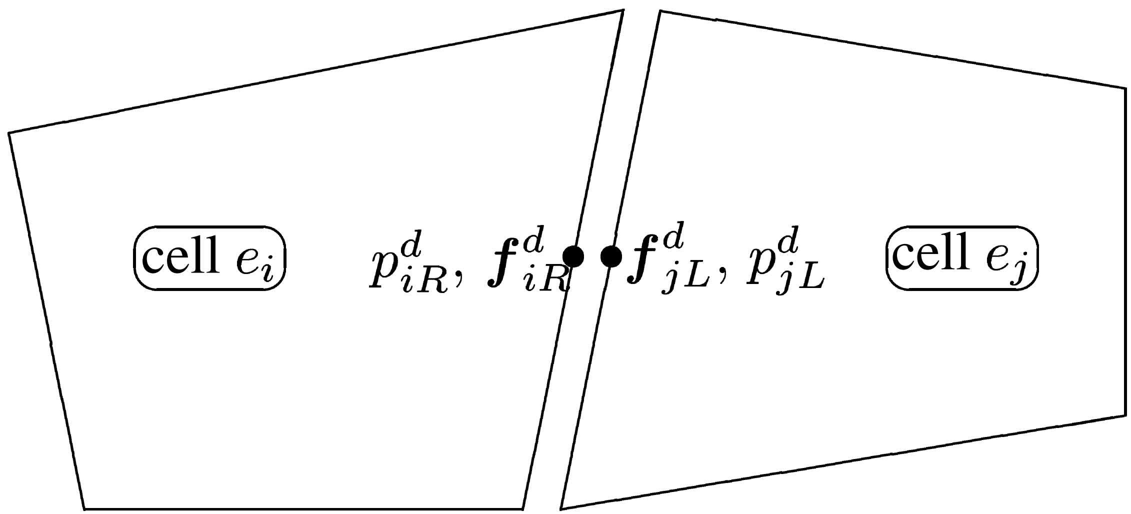
where

$$[\boldsymbol{f}^d, \boldsymbol{g}^d]_{X_i} = < \mathcal{M}_i \boldsymbol{f}_i^d, \boldsymbol{g}_i^d >$$

and $\boldsymbol{f}_i^d = (f_i^R, f_i^T, f_i^L, f_i^B)^t$. The local discretization reads

$$\begin{aligned} (\mathrm{DIV}\, \boldsymbol{f}^d)_i &= b_i^d, \\ \boldsymbol{f}_i^d &= -(\mathrm{GRAD}\, p^d)_i. \end{aligned}$$

Mimetic discretizations (4/4)



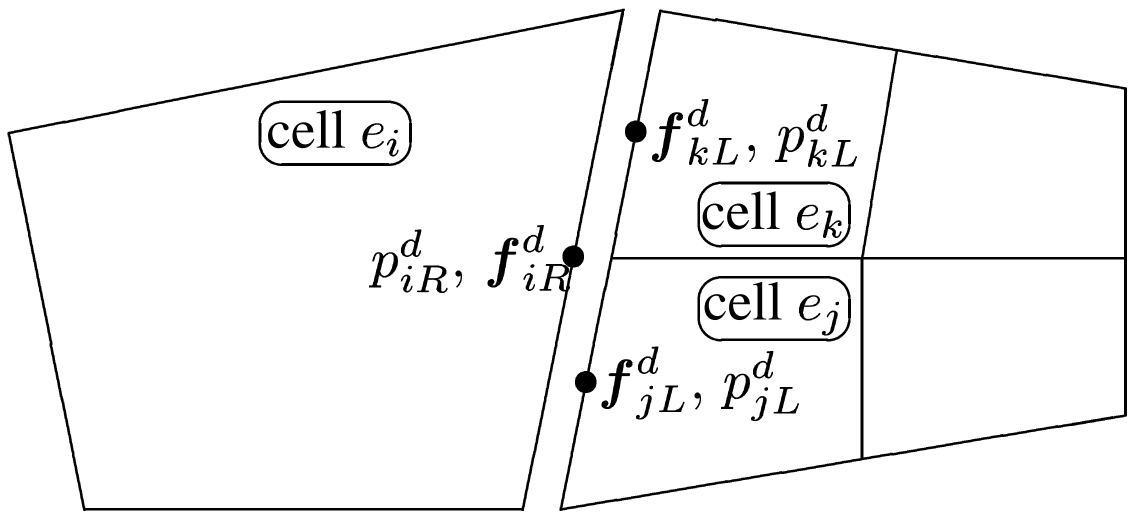
The global discretization is achieved by imposing the continuity of fluxes

$$f^d_{iR} = -f^d_{jL}$$

and interface intensities

$$p^d_{iR} = p^d_{jL}.$$

AMR grids (1/3)



The global discretization is achieved by imposing the continuity of fluxes

$$f^d_{iR} = -f^d_{jL} = -f^d_{kL}$$

and interface intensities

$$|l_{iR}|\, p^d_{iR} = |l_{jL}|\, p^d_{jL} + |l_{kL}|\, p^d_{kL}.$$

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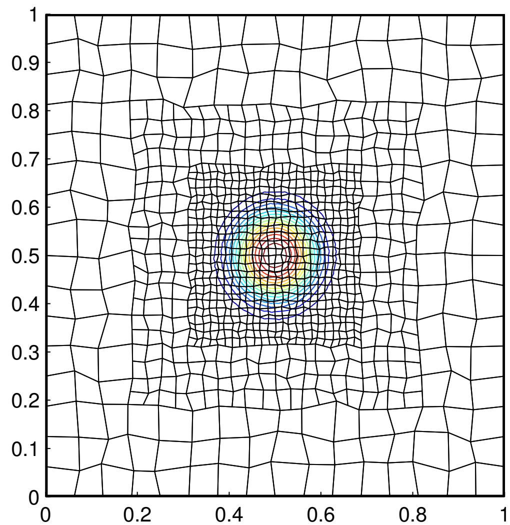
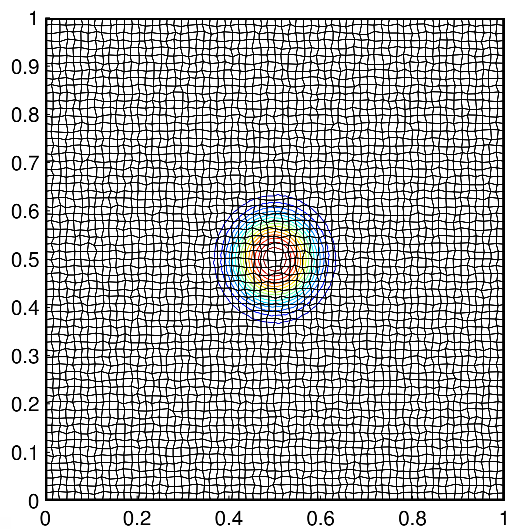
AMR grids (2/3)

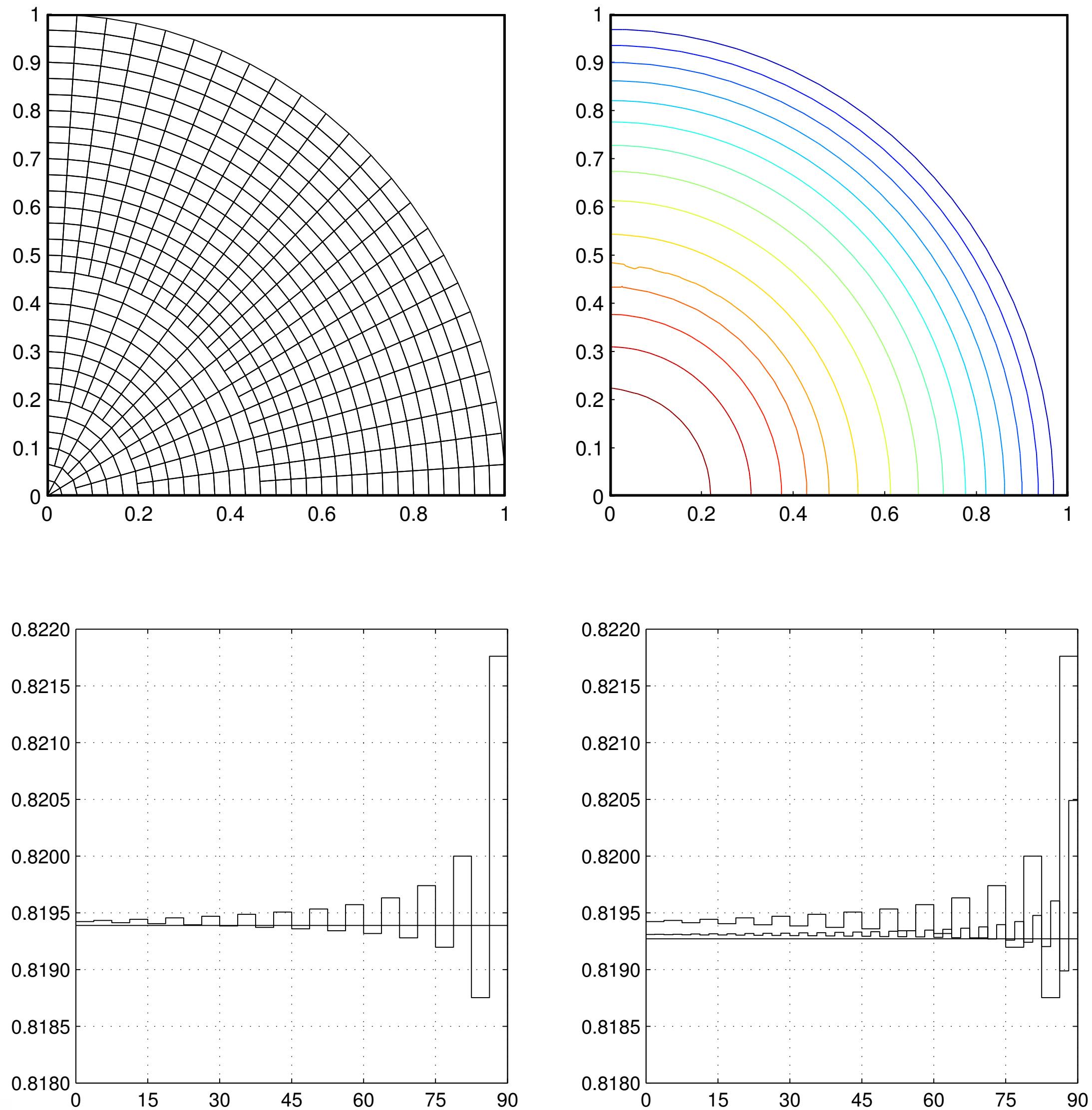




l	N	ϵ_p	$\varepsilon_{\boldsymbol{f}}$
	AMR grids		
0	256	7.00e-2	8.18e-2
1	556	1.64e-2	3.42e-2
2	988	3.74e-3	1.74e-2
3	3952	9.96e-4	7.57e-3
4	<u>15808</u>	2.40e-4	3.79e-3
	Uniform grids		
0	256	7.00e-2	8.18e-2
1	1024	1.79e-2	3.40e-2
2	4096	3.91e-3	1.62e-2
3	16384	9.44e-4	7.30e-3
4	<u>65536</u>	2.32e-4	3.76e-3

$$p(x,\, y) = 1 - \tanh\left(\frac{(x-0.5)^2 + (y-0.5)^2}{0.01}\right).$$

AMR grids (3/3)



Spherically symmetric problem in $r - z$ coordinates with the exact solution:

$$p(R) = \frac{553}{640} - \frac{R^2}{6} - \frac{R^4}{20}$$

when $R < 0.5$ and

$$p(R) = \frac{101}{120} - \frac{R^2}{12} - \frac{R^4}{40}$$

when $0.5 < R < 1$.

Conclusion (1/2)

- we proved convergence of mimetic discretizations for the linear diffusion equation; the convergence rate is optimal;
- the mimetic discretizations based on the SO methodology and FE methods are closely related for the case of triangular (or quadrilateral) conformal meshes and diffusion problems;
- the above relationships are extended to AMR triangular and quadrilateral meshes;
- the numerical experiments on general polygonal meshes show the optimal convergence rate for mimetic discretizations;
- superconvergence error estimates on triangular and quadrilateral meshes can be proved using the relationships mentioned above.

Conclusion (2/2)

References

1. M.Berndt, K.Lipnikov, D.Moulton and M.Shashkov. Convergence of mimetic finite difference discretizations of the diffusion equation, *J. Numer. Math.* (2001) **9**, No. 4, 265-284.
2. K.Lipnikov, J.Morel and M.Shashkov. Mimetic finite difference methods for diffusion equations on non-orthogonal AMR meshes, submitted to J. Comp. Physics.